Asymptotics (漸近分析)

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- **Problem solving:** (1) Recurrence of g_n:
  (2) Equation of $G(z)$: $G(z)e^{G(z)} = z$
  (3) Solve $G(z)$: $G(z) = \sqrt{(1-z)(1-3z)}$
  (4) Expand $G(z) \Leftrightarrow g_n = e^{H_n}, \frac{n+1}{n}H_n^2, \sum_{0\le k\le n}\binom{3n}{k} \sim 2\binom{3n}{n}$
  $= \binom{3n}{n}\left[2-\frac{4}{n}+O\left(\frac{1}{n^2}\right)\right]$

- $f(x) = O(g(x)),\ x \in S \Leftrightarrow \exists C,\ |f(x)| \le C|g(x)|,\ \forall x \in S$
  (1) $f(x) = O(g(x)),\ (x \to a)$ $\quad S = B(a,\varepsilon) = (a-\varepsilon, a+\varepsilon)$ $\quad a$, ε, ε
  (2) $f(n) = O(g(n)),\ (n \to \infty)$ $\quad S = (N_0, \infty)$ $\quad N_0$, ∞

- **Example** (1) $f(x) = a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 = \begin{cases} O(x^3) & (x\to 0) \\ O(x^6) & (x \to \infty)\end{cases}$ $\quad (x = \frac{1}{n})$
  (2) $g(n) = \frac{a_3}{n^3} + \frac{a_4}{n^4} + \frac{a_5}{n^5} + \frac{a_6}{n^6} = O\left(\frac{1}{n^3}\right) \quad (n \to \infty)$

- **Remarks** (1) $f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$
  (2) $f(n) = \Theta(g(n)) \Leftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \end{cases}$
  $(f(n) \asymp g(n))$
  $(f(n) \prec g(n))$ $\quad f(n) = o(g(n)) \Leftrightarrow \lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$
  $f(n) \sim g(n) \Leftrightarrow \lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$
  $\Leftrightarrow f(n) = g(n) + o(g(n))$

- **Hierarchy:** $1 \prec \log\log n \prec \log n \prec n^2 \prec n^3 \prec n^{\log n} \prec 2^n \prec n^n \prec 2^{2^n} \prec \cdots$

- **定理** (1) $O(f(n)) + O(g(n)) = O(|f(n)| + |g(n)|)$ $\quad$ (2) $cf(n) = O(f(n))$ (含 $c=1$)
  (3) $O(f(n)) \cdot O(g(n)) = O(f(n)\,g(n))$ $\quad$ (4) $O(O(f(n))) = O(f(n))$
  (5) $O(f(n)) - O(g(n)) = O(|f(n)| + |g(n)|)$

Table 452 Asymptotic approximations, valid as $n \to \infty$ and $z \to 0$.

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right). \quad (9.28)$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right). \quad (9.29)$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + O(z^5). \quad (9.32)$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + O(z^5). \quad (9.33)$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + O(z^5). \quad (9.34)$$

$$(1+z)^\alpha = 1 + \alpha z + \binom{\alpha}{2}z^2 + \binom{\alpha}{3}z^3 + \binom{\alpha}{4}z^4 + O(z^5). \quad (9.35)$$

(3.13)

• **Example 1** $\quad W = \left\lfloor \frac{n}{K} \right\rfloor + \frac{1}{2}K^2 + \frac{5}{2}K - 3 \qquad K = \lfloor \sqrt[3]{n} \rfloor = n^{\frac13} + O(1) = n^{\frac13}\left(1 + O(n^{-\frac13})\right)$

$O(n^{\frac13})$

$$= \frac{n}{n^{\frac13}\left(1+O(n^{-\frac13})\right)} + O(1) + \frac12\left(n^{\frac13}+O(1)\right)^2 + \frac52\left(n^{\frac13}+O(1)\right) - 3$$

$$= n^{\frac23}\left(1+O(n^{-\frac13})\right) + \frac12 n^{\frac23} + O(n^{\frac13})$$

$$= \underline{\underline{\frac32 n^{\frac23} + O(n^{\frac13})}}$$

• **Example 2** $\quad S_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} = \sum_{1\le k\le n} \frac{1}{n^2+k}$

$O\left(\frac{1}{n^4}\right)$

$$= \sum_{1\le k\le n} \frac{1}{n^2\left(1+\frac{k}{n^2}\right)} \qquad \frac{n^4}{n^8} + \cdots$$

$$= \sum_{1\le k\le n} \frac{1}{n^2}\left(1 - \frac{k}{n^2} + \frac{k^2}{n^4} - \frac{k^3}{n^6} + \frac{k^4}{n^8} + \cdots\right)$$

$$= \frac{n}{n^2} - \frac{1}{n^4}\frac{n(n+1)}{2} + \frac{1}{n^6}\frac{n(n+1)(2n+1)}{6} + O\left(\frac{1}{n^4}\right)$$

$$= \underline{\underline{\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + O\left(\frac{1}{n^4}\right)}}$$

$O\left(\frac{1}{n^7}\right)$

$$S_n = H_{n^2+n} - H_{n^2}$$

$$= \ln(n^2+n) + \gamma + \frac{1}{2(n^2+n)} - \frac{1}{12(n^2+n)^2} + \frac{1}{120(n^2+n)^4} + \cdots \qquad = O\left(\frac{1}{n^8}\right)$$

$$- \left[\ln n^2 + \gamma + \frac{1}{2n^2} - \frac{1}{12n^4} + \frac{1}{120n^8} + \cdots\right]$$

$$\ln(n^2+n) = \ln n^2 + \ln\left(1+\frac1n\right)$$

$$\frac{1}{2(n^2+n)} = \frac{1}{2n^2\left(1+\frac1n\right)}$$

$$\frac{-1}{12(n^2+n)^2} = \frac{-1}{12n^4}\left(1+\frac1n\right)^{-2} = \frac{-1}{12n^4}\left(1 + \binom{-2}{1}\frac1n + \binom{-2}{2}\frac{1}{n^2} + \cdots\right)$$

$$= \left(\ln n^2 + \frac1n - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \frac{1}{6n^6}\right) + \gamma + O\left(\frac{1}{n^7}\right)$$

$$+ \left(+\frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - \frac{1}{2n^5} + \frac{1}{2n^6}\right)$$

$$+ \left(-\frac{1}{12n^4} + \frac{1}{6n^5} - \frac{1}{4n^6}\right)$$

$$- \left[\ln n^2 + \gamma + \frac{1}{2n^2} \qquad - \frac{1}{12n^4}\right]$$

$$= \underline{\underline{\frac1n - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{4n^4} - \frac{2}{15n^5} + \frac{1}{12n^6} + O\left(\frac{1}{n^7}\right)}}$$

驗算: $\quad S_4 = \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} \qquad = 0.2170107$

$$S_4 = \frac14 - \frac{1}{2\cdot 4^2} - \frac16\,\frac{1}{4^3} - \frac{2}{15}\,\frac{1}{4^5} + \frac{1}{12}\,\frac{1}{4^6} + \frac14\,\frac{1}{4^4} = 0.2170125$$

$$\text{Error} = 0.0000018$$

$$\frac{1}{4^7} = 0.0000610$$

NAN PAO

3 tricks

- **Example 3** (Peturbation) $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{a}{n}+\frac{b}{n^2}+O\left(\frac{1}{n^3}\right)\right)$

$$n! = n(n-1)! = n\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}\left(1+\frac{a}{n-1}+\frac{b}{(n-1)^2}+O\left(\frac{1}{(n-1)^3}\right)\right)$$

$$\sqrt{n-1} = n^{\frac{1}{2}}\left(1-\frac{1}{n}\right)^{\frac{1}{2}} = n^{\frac{1}{2}}\left(1-\frac{1}{2}\frac{1}{n}-\frac{1}{8}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right) \qquad O\left(\frac{1}{n^3}\right)$$

$$\frac{a}{n-1} = \frac{a}{n}\frac{1}{1-1/n} = \frac{a}{n}\left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)$$

$$\frac{b}{(n-1)^2} = \frac{b}{n^2}\left(1-\frac{1}{n}\right)^{-2} = \frac{b}{n^2}\left(1 \qquad +O\left(\frac{1}{n}\right)\right)$$

$$O\left(\frac{1}{(n-1)^3}\right) := O\left(\frac{1}{n^3}\left(1-\frac{1}{n}\right)^{-3}\right) = O\left(\frac{1}{n^3}\right)$$

$$\begin{aligned}
(n-1)^{n-1} &= n^{n-1}\left(1-\frac{1}{n}\right)^{n-1} = n^{n-1}e^{(n-1)\ln\left(1-\frac{1}{n}\right)} \\
&= n^{n-1}e^{(n-1)\left(-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{3n^3}+O\left(\frac{1}{n^4}\right)\right)} \\
&= n^{n-1}e^{-1+\frac{1}{2n}+\frac{1}{6n^2}+O\left(\frac{1}{n^3}\right)} = n^{n-1}e^{-1}e^{\frac{1}{2n}+\frac{1}{6n^2}+O\left(\frac{1}{n^3}\right)} \\
&= n^{n-1}e^{-1}\left(1+\left(\frac{1}{2n}+\frac{1}{6n^2}\right)+\frac{1}{2!}\left(\frac{1}{2n}+\frac{1}{6n^2}\right)^2+O\left(\frac{1}{n^3}\right)\right) \\
&= n^{n-1}e^{-1}\left(1+\frac{1}{2n}+\frac{7}{24}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right)
\end{aligned}$$

$$\begin{aligned}
n(n-1)! &= n\sqrt{2\pi n}\left(1-\frac{1}{2n}-\frac{1}{8n^2}+O\left(\frac{1}{n^3}\right)\right)\frac{1}{e^{n-1}}\frac{n^{n-1}}{e}\left(1+\frac{1}{2n}+\frac{7}{24n^2}+O\left(\frac{1}{n^3}\right)\right)\left(1+\frac{a}{n}+\frac{a+b}{n^2}+O()\right) \\
&= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\underline{\frac{a}{n}}+\underline{\frac{a+b-1/12}{n^2}}+O\left(\frac{1}{n^3}\right)\right) \Rightarrow \begin{cases} a = \frac{1}{12} \\ b = \frac{1}{288}, \quad O\left(\frac{1}{n^4}\right) \end{cases}
\end{aligned}$$

- **Example 4** (Bootstrapping) $f(n)e^{f(n)} = n$ (de Bruijin)

$$f(n) = \ln n - \ln f(n)$$

(1) $f(n) > 1 \quad (n>e)$ 否则 $\underset{\le 1}{f(n)} + \underset{\le 0}{\ln f(n)} = \underset{>1}{\ln n}$

(2) $f(n) = O(\ln n)$

(3) $f(n) = \ln - \ln O(\ln n) = \ln n + O(\ln\ln n)$

(4)
$$\begin{aligned}
f(n) &= \ln n - \ln\left(\underline{\ln n + O(\ln\ln n)}\right) \\
&= \ln n - \ln\left(\ln n\left(1+O\left(\frac{\ln\ln n}{\ln n}\right)\right)\right) \\
&= \ln n - \ln\ln n - \ln\left(1+O\left(\frac{\ln\ln n}{\ln n}\right)\right) \\
&= \underline{\underline{\ln n - \ln\ln n + O\left(\frac{\ln\ln n}{\ln n}\right)}}
\end{aligned}$$

• Example 5 (Trading tails) $L_n = \sum_{k\ge 0} \frac{\ln(n+2^k)}{k!}$ $\quad O\left(\frac{1}{n^3}\right)$

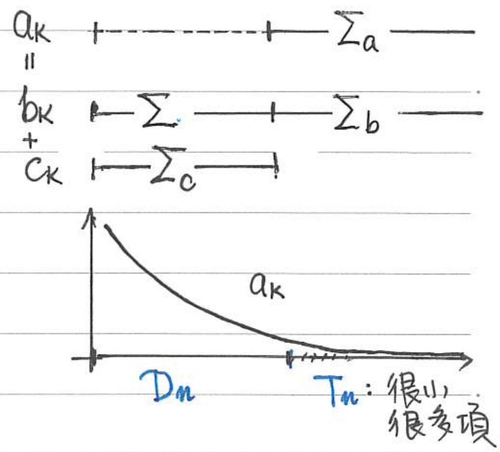
$$L_n = \sum_{k\in D_n\cup T_n} a_k(n) = \sum_{k\in D_n} a_k + \sum_{k\in T_n} a_k$$

$$= \sum_{k\in D_n}\left(b_k + O(c_k)\right) + \sum_{k\in T_n} a_k$$

$$= \sum_{k\in D_n} b_k + \sum_{k\in D_n} O(c_k) + \sum_{k\in T_n} a_k$$

$$= \sum_{k\ge 0} b_k - \sum_{k\in T_n} O(c_k) + \sum_{k\in D_n} O(c_k) + \sum_{k\in T_n} a_k$$

$$= \Sigma \quad - \Sigma_c \quad + \Sigma_b \quad + \Sigma_a$$



$$\Sigma_c = \sum_{0\le k\le \lfloor \lg n\rfloor} \frac{\frac{8^k}{n^3}}{k!} \le \frac{1}{n^3}\sum_{k\ge 0}\frac{8^k}{k!} = \frac{e^8}{n^3} = O\left(\frac{1}{n^3}\right)$$

$$\begin{cases} D_n = \{0, 1, \dots, \lfloor \lg n\rfloor - 1\} \\ T_n = \{\lfloor \lg n \rfloor, \dots\} \end{cases} \quad 2^k \le 2^{\lfloor \lg n\rfloor - 1} \le 2^{\lg n - 1} = \frac{n}{2}$$

$$\frac{2^k}{n} \le \frac{1}{2}$$

$$k \le \lg n - 1$$

$$|\Sigma_b| = \left|\sum_{k\ge \lfloor \lg n\rfloor} \frac{\ln n + \frac{2^k}{n} - \frac{4^k}{2n^2}}{k!}\right|$$

$$< \sum_{k\ge\lfloor \lg n\rfloor} \frac{\ln n + 2^k + 4^k}{k!} \qquad (\alpha = \lfloor \lg n \rfloor)$$

$$\le 3\sum_{k\ge\alpha} \frac{4^k}{k!}$$

$$= 3\left(\frac{4^\alpha}{\alpha!} + \frac{4^{\alpha+1}}{(\alpha+1)!} + \frac{4^{\alpha+2}}{(\alpha+2)!} + \cdots\right)$$

$$\le \frac{3\,4^\alpha}{\alpha!}\underbrace{\left(1 + \frac{4}{1} + \frac{4^2}{2!} + \cdots\right)}_{= e^4}$$

$$\le \frac{3\,4^{\lg n}}{\lfloor \lg n\rfloor !} = O\left(\frac{1}{n^3}\right)$$

$$\begin{cases} a_k = \dfrac{\ln(n+2^k)}{k!} \\ b_k = \dfrac{\ln n + \frac{2^k}{n} - \frac{4^k}{2n^2}}{k!} \\ c_k = \dfrac{\frac{8^k}{n^3}}{k!} \end{cases}$$

$$\ln(n+2^k) = \ln n\left(1+\frac{2^k}{n}\right) = \ln n + \ln\left(1+\frac{2^k}{n}\right)$$

$$= \ln n + \frac{2^k}{n} - \frac{4^k}{2n^2} + \frac{8^k}{3n^3} - \frac{16^k}{4n^4} + \cdots$$

$$= \ln n + \frac{2^k}{n} - \frac{4^k}{2n^2} + O\left(\frac{8^k}{n^3}\underbrace{\left(\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \cdots\right)}_{\le 1 + \frac{1}{2} + \frac{1}{4} + \cdots}\right)$$

$$= \ln n + \frac{2^k}{n} - \frac{4^k}{2n^2} + O\left(\frac{8^k}{n^3}\right)$$

$$\left(x = \frac{2^k}{n} \le \frac{1}{2}\right)$$

$$\Sigma_a = \sum_{k\ge\alpha} \frac{\ln(n+2^k)}{k!} \le \sum_{k\ge\alpha}\frac{\ln(n2^k)}{k!} = \sum_{k\ge\alpha}\frac{\ln n + \ln 2^k}{k!} = O\left(\frac{1}{n^3}\right)$$

$$\therefore\ L_n = \sum_{k\ge 0} b_k + O\left(\frac{1}{n^3}\right)$$

$$= \sum_{k\ge 0}\frac{\ln n + \frac{2^k}{n} - \frac{4^k}{2n^2}}{k!} + O\left(\frac{1}{n^3}\right)$$

$$= e\ln n + \frac{e^2}{n} - \frac{e^4}{2n^2} + O\left(\frac{1}{n^3}\right)$$

Bernoulli numbers (6.5, pp283, pp367)

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• 定理: $S_m(n) = 0^m + 1^m + 2^m + \cdots + (n-1)^m = \frac{1}{m+1}\left[\binom{m+1}{0}B_0 n^{m+1} + \binom{m+1}{1}B_1 n^m + \cdots + \binom{m+1}{m}B_m n\right]$

$1 \qquad S_0(n) = 0^0 + 1^0 + 2^0 + \cdots + (n-1)^0 = n$

$\frac{z}{1!} \qquad S_1(n) = 0^1 + 1^1 + 2^1 + \cdots + (n-1)^1 = \frac{1}{2}n^2 - \frac{1}{2}n$

$\frac{z^2}{2!} \qquad S_2(n) = 0^2 + 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$

$\frac{z^3}{3!} \qquad S_3(n) = 0^3 + 1^3 + 2^3 + \cdots + (n-1)^3 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$

$\frac{z^4}{4!} \qquad S_4(n) = 0^4 + 1^4 + 2^4 + \cdots + (n-1)^4 = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$\frac{z^5}{5!} \qquad S_5(n) = 0^5 + 1^5 + 2^5 + \cdots + (n-1)^5 = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$

$\frac{z^6}{6!} \qquad S_6(n) = 0^6 + 1^6 + 2^6 + \cdots + (n-1)^6 = \frac{1}{7}n^7 - \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

$\vdots$

$$S_0(n) = \frac{1}{1}\left[n\right]$$
$$S_1(n) = \frac{1}{2}\left[n^2 + \binom{2}{1}\left(-\frac{1}{2}\right)n\right]$$
$$S_2(n) = \frac{1}{3}\left[n^3 + \binom{3}{1}\left(-\frac{1}{2}\right)n^2 + \binom{3}{2}\frac{1}{6}n\right]$$
$$S_3(n) = \frac{1}{4}\left[n^4 + \binom{4}{1}\left(-\frac{1}{2}\right)n^3 + \binom{4}{2}\frac{1}{6}n^2\right]$$
$$S_4(n) = \frac{1}{5}\left[n^5 + \binom{5}{1}\left(-\frac{1}{2}\right)n^4 + \binom{5}{2}\frac{1}{6}n^3 + \binom{5}{4}\left(-\frac{1}{30}\right)n\right]$$
$$S_5(n) = \frac{1}{6}\left[n^6 + \binom{6}{1}\left(-\frac{1}{2}\right)n^5 + \binom{6}{2}\frac{1}{6}n^4 + \binom{6}{4}\left(-\frac{1}{30}\right)n^2\right]$$
$$S_6(n) = \frac{1}{7}\left[n^7 + \binom{7}{1}\left(-\frac{1}{2}\right)n^6 + \binom{7}{2}\frac{1}{6}n^5 + \binom{7}{4}\left(-\frac{1}{30}\right)n^3 + \binom{7}{6}\frac{1}{42}n\right]$$

$$\frac{z^m}{m!}S_m(n) = 0^m + 1^m + 2^m + \cdots + (n-1)^m \overset{?}{=} \frac{1}{m+1}\left[\binom{m+1}{0}B_0 n^{m+1} + \binom{m+1}{1}B_1 n^m + \binom{m+1}{2}B_2 n^{m-1} + \cdots + \binom{m+1}{m}B_m n\right]$$

$\vdots$ $\quad \left[n=1 \Rightarrow B_{m+1} + \delta_{m=0} = \binom{m+1}{0}B_0 + \binom{m+1}{1}B_1 + \cdots + \binom{m+1}{m}B_m + \binom{m+1}{m+1}B_{m+1}\right]$

$$\sum_{m\geq 0}\frac{z^m}{m!}S_m(n) = \sum_{m\geq 0}\frac{z^m}{m!}\left(0^m + 1^m + 2^m + \cdots + (n-1)^m\right) \qquad (n \leftarrow m+1)$$
$$= 1 + e^z + e^{2z} + \cdots + e^{(n-1)z} = \frac{e^{nz}-1}{e^z-1} = \frac{e^{nz}-1}{z}B(z)$$
$$= \left(B_0 + B_1\frac{z}{1!} + B_2\frac{z^2}{2!} + B_3\frac{z^3}{3!} + \cdots\right)\left(\frac{n}{1!} + \frac{n^2}{2!}z + \frac{n^3}{3!}z^2 + \frac{n^4}{4!}z^3 + \cdots\right)$$

$\Rightarrow$
$$S_0(n) = B_0 n$$
$$S_1(n) = \left(\frac{B_0}{0!}\frac{n^2}{2!} + \frac{B_1}{1!}\frac{n}{1!}\right) = \frac{1}{2}\left[\binom{2}{0}B_0 n^2 + \binom{2}{1}B_1 n\right]$$
$$S_2(n) = 2!\left(\frac{B_0}{0!}\frac{n^3}{3!} + \frac{B_1}{1!}\frac{n^2}{2!} + \frac{B_2}{2!}\frac{n}{1!}\right) = \frac{1}{3}\left[\binom{3}{0}B_0 n^3 + \binom{3}{1}B_1 n^2 + \binom{3}{2}B_2 n\right]$$
$$S_3(n) = 3!\left(\frac{B_0}{0!}\frac{n^4}{4!} + \frac{B_1}{1!}\frac{n^3}{3!} + \frac{B_2}{2!}\frac{n^2}{2!} + \frac{B_3}{3!}\frac{n}{1!}\right) = \frac{1}{4}\left[\binom{4}{0}B_0 n^4 + \binom{4}{1}B_1 n^3 + \binom{4}{2}B_2 n^2 + \binom{4}{3}B_3 n\right]$$

$$S_m(n) = m!\left[\frac{B_0}{0!}\frac{n^{m+1}}{(m+1)!} + \frac{B_1}{1!}\frac{n^m}{m!} + \frac{B_2}{2!}\frac{n^{m-1}}{(m-1)!} + \cdots + \frac{B_m}{m!}\frac{n}{1!}\right]$$
$$= \frac{1}{m+1}\left[\binom{m+1}{0}B_0 n^{m+1} + \binom{m+1}{1}B_1 n^m + \binom{m+1}{2}B_2 n^{m-1} + \cdots + \binom{m+1}{m}B_m n\right]$$

$$\begin{cases} B_1(x) = B_0 x + B_1 = x - \frac{1}{2} \\ B_2(x) = B_0 x^2 + \binom{2}{1} B_1 x + \binom{2}{2} B_2 = x^2 - x + \frac{1}{6} \end{cases}$$

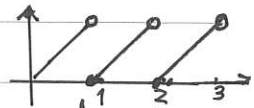
n	0	1	2	3	4	5	6	7	8	9
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0

- Bernouill #: $B_n + \delta_{n=1} = \sum_{0\le k\le n} \binom{n}{k} B_k$

- $B(z) = \sum_{n\ge 0} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}$ $\quad\because \sum_{n\ge0} B_n \frac{z^n}{n!} + z = \sum_{n\ge 0}\left(\sum_{0\le k\le n}\binom{n}{k} B_k 1^{n-k}\right)\frac{z^n}{n!}$

  - $B(z) + \frac{z}{2} = \frac{z}{2}\,\frac{e^z+1}{e^z-1}$ (偶函數) $(B_3 = B_5 = \cdots = 0)$ $\qquad B(z) + z = B(z)e^z$

- Bernouill poly: $B_m(x) = \binom{m}{0} B_0 x^m + \binom{m}{1} B_1 x^{m-1} + \binom{m}{2} B_2 x^{m-2} + \cdots + \binom{m}{m} B_m$



(0) $S_m(n) = \frac{1}{m+1}\left(B_{m+1}(n) - B_{m+1}(0)\right)$

period $=1$

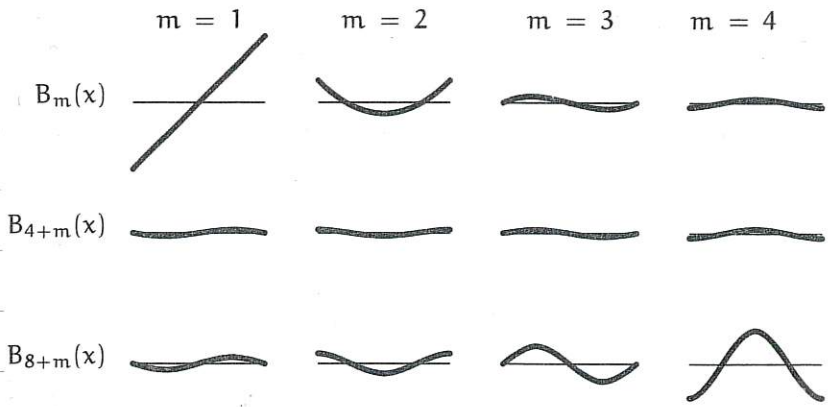
(1) $B_m(0) = B_m = B_m(1)$ $(m\ge 2)$ $\Rightarrow B_m(\{x\})$ conti $\qquad \{x\} = x - \lfloor x \rfloor$, $\lim_{x\to 1^-} B_m(\{x\}) = B(1) = B(0) = B_m(\{1\})$

(2) $B_m'(x) = m B_0 x^{m-1} + m\binom{m-1}{1} B_1 x^{m-2} + m\binom{m-1}{2} B_2 x^{m-3} + \cdots + m\binom{m-1}{m-1} B_{m-1}$
$= m B_{m-1}(x)$

(3) $B_{2m}(x)$, $0\le x\le 1$, 在 $x=0$ 或 $\frac{1}{2}$ 取 max/min

(4) $B_{2m}(\frac{1}{2}) = (2^{1-2m} - 1) B_{2m}$ (Ex.17)
$\Rightarrow |B_{2m}(\{x\})| \le |B_{2m}|$

(5) $H_\infty^{(2m)} = (-1)^{m-1}\frac{(2\pi)^{2m}}{2(2m)!} B_{2m}$
$\Rightarrow \frac{|B_{2m}|}{(2m)!} = O\left(\frac{1}{(2\pi)^{2m}}\right)$



- Euler summation formula

$$\boxed{\sum\nolimits_a^b f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m,} \qquad \boxed{R_m = (-1)^{m+1}\int_a^b \frac{B_m(\{x\})}{m!} f^{(m)}(x)\,dx}$$

$$\sum_{1\le k<n} f(k) = \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \frac{B_2}{2!} f'(x)\Big|_1^n + \frac{B_3}{3!} f^{(2)}(x)\Big|_1^n + \cdots + \frac{B_m}{m!} f^{(m-1)}(x)\Big|_1^n + R_m$$

證: $\int_1^n B_1(\{x\}) f'(x)\,dx = \sum_{1\le k<n}\int_k^{k+1} \overbrace{(\{x\} - \tfrac{1}{2})}^{x-k-\frac{1}{2}} f'(x)\,dx = \sum_{1\le k<n}\left[(x-k-\tfrac12)f(x)\Big|_k^{k+1} - \int_k^{k+1} f(x)\,dx\right]$

$= \sum_{1\le k<n}\left[\tfrac12 f(k+1) + \tfrac12 f(k) - \int_k^{k+1} f(x)\,dx\right] = \sum_{1\le k<n} f(k) + \underbrace{\tfrac12}_{-B_1}\left(f(n) - f(1)\right) - \int_1^n f(x)\,dx$

(m=1) $\sum_{1\le k<n} f(k) = \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \int_1^n B_1(\{x\}) f'(x)\,dx$ $\parallel$ $\int_1^n f'(x)\, d\,\frac{B_2(\{x\})}{2}$

(m=2) $= \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \frac{B_2}{2!} f'(x)\Big|_1^n - \frac{1}{2!}\int_1^n B_2(\{x\}) f^{(2)}(x)\,dx$

(m=3) $= \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \frac{B_2}{2!} f'(x)\Big|_1^n + \frac{\overset{=0}{B_3}}{3!} f^{(2)}(x)\Big|_1^n + \frac{1}{3!}\int_1^n B_3(\{x\}) f^{(3)}(x)\,dx$

(m=m-1) $= \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \frac{B_2}{2!} f'(x)\Big|_1^n + \cdots + \frac{B_{m-1}}{(m-1)!} f^{(m-2)}(x)\Big|_1^n + \frac{(-1)^m}{(m-1)!}\int_1^n B_{m-1}(\{x\}) f^{(m-1)}(x)\,dx$

(m=m) $= \int_1^n f(x)\,dx + B_1 f(x)\Big|_1^n + \frac{B_2}{2!} f'(x)\Big|_1^n + \cdots + \frac{\cancel{(-1)^m} B_m}{m!} f^{(m-1)}(x)\Big|_1^n + \frac{(-1)^{m+1}}{m!}\int_1^n B_m(\{x\}) f^{(m)}(x)\,dx$

K	0	1	2	3	4	5
B_K	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0

- **Example 1**

$$\sum_{0\le k<n} k^3 = \int_0^n x^3 dx + B_1 x^3 \Big|_0^n + \frac{B_2}{2} 3x^2 \Big|_0^n + \frac{B_3}{3!} 6x \Big|_0^n + \frac{B_4}{4!} 6 \Big|_0^n$$

$$= \frac{1}{4} n^4 + B_1 n^3 + \frac{B_2}{2!} 3n^2 + \frac{B_3}{3!} 6n$$

$$= \frac{1}{4}\left[\binom{4}{0} B_0 n^4 + \binom{4}{1} B_1 n^3 + \binom{4}{2} B_2 n^2 + \binom{4}{3} B_3 n\right]$$

- **Example 2**

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right)$$

$$f(x) = \frac{1}{x}, \quad f'(x) = \frac{-1}{x^2}, \quad \cdots, \quad f^{(k)}(x) = (-1)^k k! \frac{1}{x^{k+1}}$$

證:

$$H_{n-1} = \sum_1^n f(k) = \int_1^n \frac{1}{x} dx + \sum_{1\le k\le m} \frac{B_k}{k!} (-1)^{k-1} (k-1)! \left(\frac{1}{n^k} - \frac{1}{1}\right) + \frac{(-1)^{m+1}}{m!} \int_1^n B_m(\{x\}) \frac{(-1)^m m!}{x^{m+1}} dx$$

$$= \ln n + \sum_{1\le k\le m} \frac{(-1)^{k-1} B_k}{k} \frac{1}{n^k} + \sum_{1\le k\le m} \frac{(-1)^k B_k}{k} - \int_1^n \frac{B_m(\{x\})}{x^{m+1}} dx$$

$$= \ln n + \sum_{1\le k\le m} \frac{(-1)^{k-1} B_k}{k} \frac{1}{n^k} + \underline{\sum_{1\le k\le m} \frac{(-1)^k B_k}{k} - \int_1^\infty \frac{B_m(\{x\})}{x^{m+1}} dx} + \underbrace{\int_n^\infty \frac{B_m(\{x\})}{x^{m+1}} dx}_{O\left(\frac{1}{n^m}\right)}$$

$$= \ln n + \gamma + \sum_{1\le k\le m-1} \frac{(-1)^{k-1} B_k}{k} \frac{1}{n^k} + O\left(\frac{1}{n^m}\right)$$

$(m=6)$

$$H_n = n + \ln n + \gamma + B_1 \frac{1}{n} + \frac{-B_2}{2} \frac{1}{n^2} + \frac{-B_4}{4} \frac{1}{n^4} + O\left(\frac{1}{n^6}\right)$$

$$= \ln n + \gamma + \frac{1}{2n} - \frac{1}{12} \frac{1}{n^2} + \frac{1}{120} \frac{1}{n^4} + O\left(\frac{1}{n^6}\right) \quad \#$$

- **Remarks**

(1) $$\left|\int_n^\infty \frac{B_m(\{x\})}{x^{m+1}} dx\right| \le \int_n^\infty \frac{|B_m|}{x^{m+1}} dx = |B_m| \lim_{c\to\infty} \int_n^c \frac{1}{x^{m+1}} dx = \frac{|B_m|}{-m} \lim_{c\to\infty} \left[\frac{1}{x^m}\right]_n^c = O\left(\frac{1}{n^m}\right)$$

(2) $$\int_1^\infty \frac{B_m(\{x\})}{x^{m+1}} dx = A \quad \text{(同理)}$$

(3) $$\gamma = \lim_{n\to\infty} (H_{n-1} - \ln n) = \lim_{n\to\infty} \left[\sum_{1\le k\le m} \frac{(-1)^{k-1} B_k}{k} \frac{1}{n^k} + \sum_{1\le k\le m} \frac{(-1)^k B_k}{k} - A + O\left(\frac{1}{n^m}\right)\right]$$

$$\int \ln x \, dx = x \ln x - x$$

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• **Example 3** (Stirling)

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840}\frac{1}{n^3} - \frac{571}{2488320}\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)\right)$$

$$n! = e^{\ln n!} = e^{\ln n + \sum_{1 \le k < n} \ln k}, \qquad f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}$$

證：

$$\sum_{1 \le k < n} \ln k = \int_1^n \ln x\, dx + \left(-\frac{1}{2}\right)\ln n + \sum_{2 \le k \le m} \frac{B_k (-1)^{k-2}(k-2)!}{k!\, x^{k-1}}\Bigg|_1^n + \frac{(-1)^{m+1}}{m!}\int_1^n \frac{B_m(\{x\})(-1)^{m-1}(m-1)!}{x^m}\,dx$$

$$= (n\ln n - n + 1) - \frac{1}{2}\ln n + \sum_{2 \le k \le m} \frac{(-1)^k B_k}{k(k-1)}\frac{1}{n^{k-1}} + \sum_{2 \le k \le m}\frac{(-1)^{k+1}B_k}{k(k-1)} + \frac{1}{m}\int_1^n \frac{B_m(\{x\})}{x^m}\,dx$$

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n + \sum_{2 \le k \le m}\frac{(-1)^k B_k}{k(k-1)}\frac{1}{n^{k-1}} + \underline{1 + \sum_{2 \le k \le m}\frac{(-1)^{k+1}B_k}{k(k-1)} + \frac{1}{m}\int_1^\infty \frac{B_m(\{x\})}{x^m}\,dx} \underbrace{- \frac{1}{m}\int_n^\infty \cdots dx}_{O\left(\frac{1}{n^{m-1}}\right)}$$

$$= \left(n+\frac{1}{2}\right)\ln n - n + \sigma + \sum_{2 \le k \le m-1}\frac{(-1)^k B_k}{k(k-1)}\frac{1}{n^{k-1}} + O\left(\frac{1}{n^{m-1}}\right)$$

$(m=6)$

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)$$

$$n! = e^{\left(n+\frac{1}{2}\right)\ln n - n + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)} \qquad \left(e^\sigma = \sqrt{2\pi}\right)$$

$$= n^{n+\frac{1}{2}} e^{-n}\sqrt{2\pi}\left[1 + \frac{1}{12n} + \frac{1}{288n^2} + \left(\frac{-1}{360} + \frac{1}{6\cdot 12^3}\right)\frac{1}{n^3} + \left(\frac{-2}{2\cdot 12\cdot 360} + \frac{1}{4!}\frac{1}{12^4}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)\right]$$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840}\frac{1}{n^3} - \frac{571}{2488320}\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)\right]$$

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